THE METHOD OF A SMALL PARAMETER IN THE CLASSICAL STEFAN PROBLEM

A. O. Gliko and A. B. Efimov

UDC 536.24.02

It is shown that the motion of a phase interface with relatively small perturbations of the boundary condition is described by the Volterra linear integral equation. The solution is investigated using a Laplace transformation.

Let us consider the one-dimensional, two-phase, Stefan problem

$$\begin{gathered}\frac{\partial u}{\partial t}=a_1^2\frac{\partial^2 u}{\partial z^2}\quad (0<z<\zeta(t)),\ \zeta(0)=\zeta_0,\\ u(z,\ 0)=u_0(z),\ u(0,\ t)=T_0,\ u(\zeta(t),\ t)=v(\zeta(t),\ t)=T_m,\\ \frac{\partial v}{\partial t}=a_2^2\frac{\partial^2 v}{\partial z^2}\quad (\zeta(t)<z<l),\\ -\varkappa_1\left.\frac{\partial u}{\partial z}\right|_{z=\zeta(t)-0}+\varkappa_2\left.\frac{\partial v}{\partial z}\right|_{z=\zeta(t)+0}=-\lambda\rho\frac{d\zeta}{dt},\\ v(z,\ 0)=v_0(z),\ v(l,\ t)=T_l(t),\end{gathered}\tag{1}$$

$u_0(z)$ and $v_0(z)$ being the initial temperature distributions. If the temperature of the phase transition depends on the z coordinate, then $T_m = T_m(\zeta(t))$. In the general case this problem is reduced to a complex system of nonlinear integral equations [1-2]. We will consider the system (1) with the following restrictions: 1) at the initial time the boundary is at a stationary position; 2) the variation in the temperature $v(l,\ t)$ represents a small perturbation relative to the steady value,

$$v(l,\ t)=T_l(t)=T_1+\varepsilon T(t).\tag{2}$$

The initial conditions are assigned as linear functions of z and are consistent with the boundary conditions:

$$\begin{gathered}u(z,\ 0)=(T_m-T_0)\frac{z}{\zeta_0}+T_0,\\ v(z,\ 0)=(T_1-T_m)\frac{z-\zeta_0}{l-\zeta_0}+T_m,\ T_m=T_m(\zeta_0).\end{gathered}\tag{3}$$

We introduce the coordinate system connected with the phase interface [1]:

$$p=\frac{\zeta(t)-z}{\zeta(t)},\quad q=\frac{z-\zeta(t)}{l-\zeta(t)}.\tag{4}$$

In the (pqt) coordinates the system (1) is written as follows:

$$\frac{\partial u}{\partial t}=\frac{a_1^2}{\zeta^2}\frac{\partial^2 u}{\partial p^2}-\frac{(1-p)}{\zeta}\frac{d\zeta}{dt}\frac{\partial u}{\partial p}\quad (0<p<1),$$

Institute of Earth Physics, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 38, No. 2, pp. 329-335, February, 1980. Original article submitted April 5, 1979.

0022-0841/80/3802-0211$07.50

$$\frac{\partial v}{\partial t}=\frac{a_2^2}{(l-\zeta)^2}\frac{\partial^2 v}{\partial q^2}+\frac{(1-q)}{l-\zeta}\frac{d\zeta}{dt}\frac{\partial v}{\partial q}\quad(0<q<1),$$

$$\frac{\varkappa_1}{\zeta}\frac{\partial u(0,t)}{\partial p}+\frac{\varkappa_2}{l-\zeta}\frac{\partial v(0,t)}{\partial q}=-\lambda\rho\frac{d\zeta}{dt}, \tag{5}$$

$$u(p,0)=T_m(\zeta_0)-(T_m(\zeta_0)-T_0)p,\ v(q,0)=T_m(\zeta_0)+(T_1-T_m(\zeta_0))q,$$

$$u(0,t)=v(0,t)=T_m(\zeta(t)),\ u(1,t)=T_0,\ v(1,t)=T_l(t).$$

With allowance for (2), we seek the solution of the problem (5) in the form of an expansion with respect to a small parameter:

$$\zeta(t)=\zeta_0(1-\varepsilon\mu(t)),$$
$$u=u_0(p)+\varepsilon u_1(p,t),\ v=v_0(q)+\varepsilon v_1(q,t). \tag{6}$$

In addition, it is also necessary to expand the phase curve $T_m(\zeta)$ with respect to ε;

$$T_m(\zeta(t))=T_m(\zeta_0)-\varepsilon T_{m_1}\mu(t). \tag{7}$$

Substituting the expansions of the functions $\zeta(t)$, $u(p, t)$, and $v(q, t)$ into the system (5), and using the expansions (2) and (7) of the boundary conditions, we determine the zeroth approximations ζ_0, $u_0(p)$, and $v_0(q)$ with respect to ε and obtain a system for finding the first approximation, $u_1(p, t)$, $v_1(q, t)$, and $\mu(t)$.

The zeroth approximations are

$$u_0(p)=T_m(\zeta_0)-(T_m(\zeta_0)-T_0)p,\ v_0(q)=T_m(\zeta_0)+(T_1-T_m(\zeta_0))q. \tag{8}$$

The coordinate ζ_0 corresponding to the steady-state solution is determined from the transcendental equation

$$\frac{\varkappa_1}{\zeta_0}(T_m(\zeta_0)-T_0)=\frac{\varkappa_2}{l-\zeta_0}(T_1-T_m(\zeta_0)). \tag{9}$$

In the first approximation with respect to ε we obtain the system

$$\frac{\partial u_1}{\partial t}=\frac{1}{\tau_0}\frac{\partial^2 u_1}{\partial p^2}-(T_m(\zeta_0)-T_0)(1-p)\frac{d\mu}{dt}\quad(0<p<1),$$

$$\frac{\partial v_1}{\partial t}=\frac{1}{\tau_1}\frac{\partial^2 v}{\partial q^2}-(T_1-T_m(\zeta_0))\frac{\zeta_0}{l-\zeta_0}(1-q)\frac{d\mu}{dt}\quad(0<q<1), \tag{10}$$

$$\frac{-\varkappa_1 l(T_m(\zeta_0)-T_0)}{\zeta_0(l-\zeta_0)}\mu(t)+\frac{\varkappa_1}{\zeta_0}\frac{\partial u_1(0,t)}{\partial p}+\frac{\varkappa_2}{l-\zeta_0}\frac{\partial v_1(0,t)}{\partial q}=\lambda\rho\zeta_0\frac{d\mu}{dt},$$

$$u_1(1,t)=u_1(p,0)=v_1(q,0)=0,\ u_1(0,t)=v_1(0,t)=-T_{m1}\mu(t),$$

$$v_1(1,t)=T(t),\ \tau_0=\zeta_0^2/a_1^2,\ \tau_1=(l-\zeta_0)^2/a_2^2.$$

We seek the solutions of the first two equations of the system (10) in the form $u_1 = u_{11} + u_{12}$ and $v_1 = v_{11} + v_{12} + v_{13}$, where u_{11} and v_{11} are the solutions of the inhomogeneous equations of heat conduction for u_1 and v_1 with zero initial and boundary data, while u_{12}, v_{12}, and v_{13} are the solutions of homogeneous equations with the corresponding inhomogeneous boundary conditions. These solutions are written using the Green's function for the first boundary problem in the segment (0; 1) [3-4]:

$$u_{11}(p,t)=-2(T_m-T_0)\int_0^t\frac{d\mu}{d\tau}\sum_{n=1}^{+\infty}\frac{1}{\pi n}\exp\left[-\frac{\pi^2 n^2(t-\tau)}{\tau_0}\right]\sin\pi np\,d\tau. \tag{11}$$

Calculating the derivative $\partial u_{11}/\partial p$ by termwise differentiation of the integrand and using the definition of the Jacobi elliptic theta function Θ_3 [5],

$$\Theta_3(p,\ t) = 1 + 2\sum_{n=1}^{+\infty} \exp\left[-\pi^2 n^2 t\right] \cos 2\pi np, \tag{12}$$

we obtain

$$\frac{\partial u_{11}(p,\ t)}{\partial p} = -(T_m - T_0)\int_0^t \frac{d\mu}{d\tau}\left[\Theta_3\left(\frac{p}{2};\ \frac{t-\tau}{\tau_0}\right) - 1\right] d\tau. \tag{13}$$

We obtain the expression $\partial v_{11}/\partial q$ similarly:

$$\frac{\partial v_{11}(q,\ t)}{\partial q} = -(T_1 - T_m)\frac{\zeta_0}{l-\zeta_0}\int_0^t \frac{d\mu}{d\tau}\left[\Theta_3\left(\frac{q}{2};\ \frac{t-\tau}{\tau_1}\right) - 1\right] d\tau. \tag{14}$$

The solution of the homogeneous equation for $u_1(p,\ t)$ with the boundary conditions $u_1(0,\ t) = -T_{m1}\mu(t)$ and $u_1(1,\ t) = 0$ and a zero initial condition has the form

$$u_{12}(p,\ t) = -\frac{2\pi T_{m1}}{\tau_0}\int_0^t \mu(\tau)\sum_{n=1}^{+\infty} \exp\left[-\frac{\pi^2 n^2 (t-\tau)}{\tau_0}\right] n \sin \pi npd\tau. \tag{15}$$

Integrating the series in (15) by parts and calculating the derivative with respect to p, we obtain

$$\frac{\partial u_{12}(p,\ t)}{\partial p} = T_{m1}\int_0^t \frac{d\mu}{d\tau}\Theta_3\left(\frac{p}{2};\ \frac{t-\tau}{\tau_0}\right) d\tau. \tag{16}$$

The expression $\partial v_{12}(q,\ t)/\partial q$ looks completely similar:

$$\frac{\partial v_{12}(q,\ t)}{\partial q} = T_{m1}\int_0^t \frac{d\mu}{d\tau}\Theta_3\left(\frac{q}{2};\ \frac{t-\tau}{\tau_1}\right) d\tau\ . \tag{17}$$

Finally,

$$v_{13}(q,\ t) = -\frac{2\pi}{\tau_1}\int_0^t T(\tau)\sum_{n=0}^{+\infty} (-1)^n n \exp\left[-\frac{\pi^2 n^2 (t-\tau)}{\tau_1}\right] \sin \pi nqd\tau. \tag{18}$$

Integrating the series in (18) by parts and calculating the derivative with respect to q, we obtain

$$\frac{\partial v_{13}(q,\ t)}{\partial q} = \int_0^t \frac{dT}{d\tau}\Theta_3\left(\frac{1-q}{2};\ \frac{t-\tau}{\tau_1}\right) d\tau. \tag{19}$$

Substituting the values of the derivatives $\partial u_1(0,\ t)/\partial p$ and $\partial v_1(0,\ t)/\partial q$ into the third equation of the system (10) and using Eq. (9), we obtain an integral equation for the velocity of motion $d\mu/dt$ of the interface:

$$\tilde{T}_1\sqrt{\tau_0\tau_1}\frac{d\mu}{dt} + \int_0^t \frac{d\mu}{d\tau}\left[\tilde{T}_2\sqrt{\frac{\tau_1}{\tau_0}}\Theta_3\left(0;\ \frac{t-\tau}{\tau_0}\right) + \right.$$
$$\left. + \tilde{T}_3\Theta_3\left(0;\ \frac{t-\tau}{\tau_0}\right)\right]\partial\tau - A\int_0^t \frac{\partial T}{\partial\tau}\Theta_3\left(\frac{1}{2};\ \frac{t-\tau}{\tau_1}\right) d\tau\ . \tag{20}$$

Here

$$\tilde{T}_1=\frac{\lambda}{C_1};\ \tilde{T}_2=T_m-T_0-T_{m1};\ \tilde{T}_3=\sqrt{\frac{C_2\varkappa_1}{C_1\varkappa_2}}\left(T_m-T_0-\frac{\varkappa_2}{\varkappa_1}T_{m1}\right);$$
$$A=\sqrt{\frac{\varkappa_2 C_2}{\varkappa_1 C_1}};$$

C_1 and C_2 are the heat capacities of phases 1 and 2, respectively. We will investigate the Volterra integral equation (20) with the help of a Laplace transformation. Changing to the transforms, and considering that [6]

$$\Theta_3(p,\ t)=\frac{\operatorname{ch}(2p-1)\sqrt{s}}{\sqrt{s}\operatorname{sh}\sqrt{s}},\tag{21}$$

we obtain

$$M(s)=\frac{A\chi(s)}{\operatorname{sh}\sqrt{s\tau_1}}\,\frac{1}{\tilde{T}_1\sqrt{s\tau_0}+\tilde{T}_2\operatorname{cth}\sqrt{s\tau_0}+\tilde{T}_3\operatorname{cth}\sqrt{s\tau_1}},\tag{22}$$

where M(s) and $\chi(s)$ are the Laplace transforms of the functions $\mu(t)$ and T(t). Let us investigate the principal asymptotic forms of the solution. At "small" times ($t \ll \min(\tau_0, \tau_1)$) the behavior of $\mu(t)$ is determined by the asymptotic form M(s) at large s [6]. At large s

$$M(s)\simeq\frac{2A\exp(-\sqrt{s\tau_1})\chi(s)}{\tilde{T}_1\sqrt{s\tau_0}+\tilde{T}_2+\tilde{T}_3}.\tag{23}$$

By inverting (23) we obtain an expression describing the motion of the boundary at times $t \ll \min(\tau_0, \tau_1)$:

$$\mu(t)\simeq\frac{2A}{\tilde{T}_2+\tilde{T}_3}\int_0^t T(t)\,F'(t-\tau)\,d\tau,\tag{24}$$

where

$$F(t)=\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{\tau_1}{t}}\right)-\exp\left(\alpha^2\frac{t}{\tau_0}+\alpha\sqrt{\frac{\tau_1}{\tau_0}}\right)\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{\tau_1}{t}}+\alpha\sqrt{\frac{t}{\tau_0}}\right).$$

The quantity α equals $(\tilde{T}_2+\tilde{T}_3)/\tilde{T}_1$, and for the case of $\tau_1 \ll \tau_0/\alpha^2$ we can obtain a comparatively simple quadrature for the coordinate of the boundary, using an asymptotic expansion of the function erfc (z) at large z:

$$\mu(t)\simeq\frac{2A\sqrt{\tau_1}}{\sqrt{\pi}\tilde{T}_1\tau_0}\int_0^t T(\tau)\,\frac{\exp\left[-\dfrac{\tau_1}{4(t-\tau)}\right]}{\sqrt{t-\tau}}\,d\tau.\tag{25}$$

The motion of the boundary at large times ($t \gg \max(\tau_0, \tau_1)$) is determined by the residues of the function $M(s)e^{ts}$ at the poles closest to the origin of coordinates. Let us consider the case when $\chi(s) = T/s$ is the transform of the step function $T(t) = TH(t)$. A residue of zero gives a constant

$$M_0=\frac{AT}{\tilde{T}_2(\tau_1/\tau_0)^{1/2}+\tilde{T}_3}.$$

By analyzing the denominator of Eq. (22) we can ascertain that the next poles lie on the negative s semiaxis and their location is found from the solution of the transcendental equation

$$\tilde{T}_3\operatorname{ctg}b+\tilde{T}_2\operatorname{ctg}b\sqrt{\frac{\tau_0}{\tau_1}}+\tilde{T}_1 b\sqrt{\frac{\tau_0}{\tau_1}}=0,\ b=-i\sqrt{s\tau_1}.\tag{26}$$

We represent $\mu(t)$ in the series form

$$\mu(t) = M_0 + \sum_{n=1}^{+\infty} M_n \exp(-b_n^{*2} t/\tau_1), \tag{27}$$

$$M_n = -\frac{2A}{\sin b_n^*} \frac{1}{\tilde{T}_1 + \tilde{T}_2/\sin^2 b_n^* \sqrt{\frac{\tau_0}{\tau_1}} + \tilde{T}_3/\sin^2 b_n^*}. \tag{28}$$

In the case when $\tau_1 \ll \tau_0$ the location of the first N poles $\left(N \sim \frac{1}{10\pi} \sqrt{\frac{\tau_0}{\tau_1}}\right)$ is determined by the equation

$$b_n^* \simeq \sqrt{\frac{\tau_1}{\tau_0}} \left(1 - \frac{\tilde{T}_2}{\tilde{T}_3} \sqrt{\frac{\tau_1}{\tau_0}}\right) \pi n.$$

At times $t \gtrsim \tau_0$ we can retain only the first term of the series (27). Then we obtain the following equation describing the emergence of the phase boundary at the new steady position:

$$\mu(t) = M_0 - \frac{2A\pi^2 T}{\tilde{T}_3(1 + \tilde{T}_3/\tilde{T}_2)} \sqrt{\frac{\tau_1}{\tau_0}} \exp[-\pi^2 t/\tau_0]. \tag{29}$$

Knowing the law of motion of the boundary, we can determine the temperature distributions in each of the phases. We apply a Laplace transformation to Eqs. (13) and (16):

$$\frac{\partial U(p, s)}{\partial p} = (T_m - T_0) M(s) - (T_m - T_0 - T_{m1}) M(s) \sqrt{s\tau_0} \frac{\operatorname{ch}(1-p)\sqrt{s\tau_0}}{\operatorname{sh}\sqrt{s\tau_0}}. \tag{30}$$

Integrating over p, at large s we obtain

$$U(p, s) \simeq (T_m - T_0) M(s) p + (T_m - T_0 - T_{m1}) M(s) \exp[-p\sqrt{s\tau_0}]. \tag{31}$$

From this we get the temperature distribution in the layer $0 < z < \zeta(t)$ at small t:

$$u_1(p, t) \simeq (T_m - T_0) p\mu(t) + (T_m - T_0 - T_{m1}) \frac{p\sqrt{\tau_0}}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{p^2\tau_0}{4(t-\tau)}\right] d\tau. \tag{32}$$

Similarly, applying a Laplace transformation to Eqs. (14), (17), and (19) and integrating over q, at large s we obtain

$$V(q, s) \simeq \frac{\varkappa_1}{\varkappa_2}(T_m - T_0) M(s) q + \left(\frac{\varkappa_1}{\varkappa_2}(T_m - T_0) - T_{m1}\right) M(s) \exp(-q\sqrt{s\tau_1}) - \chi(s) \exp[-(1-q)\sqrt{s\tau_1}]. \tag{33}$$

From this we find the temperature distribution in the layer $\zeta(t) < z < l$ at small t:

$$v_1(q, t) \simeq \frac{\varkappa_1}{\varkappa_2}(T_m - T_0) q\mu(t) - \left(\frac{\varkappa_1}{\varkappa_2}(T_m - T_0) - T_{m1}\right) \times$$

$$\times \frac{q\sqrt{\tau_1}}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{q^2\tau_1}{4(t-\tau)}\right] d\tau - \frac{(1-q)}{2\sqrt{\pi}} \sqrt{\tau_1} \int_0^t \frac{T(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{(1-q)^2\tau_1}{4(t-\tau)}\right] d\tau. \tag{34}$$

In conclusion, we note that in the general case the numerical solutions of the integral equation (20) can easily be obtained by the method of successive approximations.

NOTATION

$u(z, t)$ and $v(z, t)$, temperature distributions in regions of phases 1 and 2, respectively; $\zeta(t)$, coordinate of moving phase boundary; a_i^2, $\varkappa_i$, and C_i, thermal diffusivity, thermal conductivity, and heat capacity of i-th phase (i = 1, 2); λ, latent heat of transition; ε, small parameter; $\tau_0 = \zeta_0^2/a_1^2$ and $\tau_1 = (l - \zeta_0)^2/a_2^2$, characteristic times of heating; s, complex variable in Laplace transformation.

LITERATURE CITED

1. L. I. Rubinshtein, Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz., No. 1 (1947).
2. L. I. Rubinshtein, The Stefan Problem [in Russian], Zvaigzne, Riga (1967).
3. A. V. Lykov and Yu. A. Mikhailov, The Theory of Heat and Mass Exchange [in Russian], Gosénergoizdat, Moscow–Leningrad (1963).
4. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 2nd ed., Clarendon Press, Oxford (1959).
5. H. Jeffreys and B. Swirles, Methods of Mathematical Physics, Part 3, Cambridge University Press, England (1966).
6. G. Doetsch, Guide to the Applications of Laplace Transforms, Van Nostrand, London–New York (1963).